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Lattice Models and Generalized Rogers Ramanujan Identities

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ABSTRACT

We revisit the solvable lattice models described by Andrews Baxter and Forrester and their generalizations. The expressions for the local state probabilities were shown to be related to characters of the minimal models. We recompute these local state probabilities by a different method. This yields generalized Rogers Ramanujan identities, some of which recently conjectured by Kedem et al. Our method provides a proof for some cases, as well as generating new such identities.

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Several different systems in two dimensional physics appear to be closely related. These include integrable $N = 2$ systems, rational conformal field theories (RCFT), solvable lattice models and integrable massive field theories (see, for example, ref. [1]). There are other, rather mysterious, examples of such connections. One is the appearance of the characters of the fixed point conformal field theory in the expression for the local state probabilities (LSP) on the lattice [2, 3, 4]. Presumably, the explanation for this phenomenon lies in the context of the aforementioned correspondence between lattice models and RCFT. Another intriguing observation is the connection between generalized Rogers–Ramanujan (GRR) identities for the characters of RCFT, and the thermodynamic Bethe ansatz (TBA) equations for the massive perturbation of the theory [5].

These observations remain largely unexplained. Our purpose here is to resolve a little part of the puzzle. We do this by exploring yet another connection. It is shown that the very same generalized Rogers–Ramanujan identities naturally arise as an expression for the local state probabilities of the corresponding lattice models. This closes a circle, as those LSP are given in terms of the RCFT characters, which, in turn, can be expressed as GRR. This provides a systematic derivation for these identities, along with a proof of their validity.

For the sake of concreteness the following picture is conjectured. There is a one to one correspondence between RCFT, solvable lattice models, GRR, and TBA. Our purpose here is to explore the connection between GRR and lattice models. In this context we conjecture that every solvable lattice model leads through the LSP to a GRR which is the character of the fixed point RCFT. Indeed this is how Rogers–Ramanujan identities first arose in physics, in the expressions for the LSP of the hard hexagon model [6]. Specifically, we explore here the coset models $SU(2)_{k-m} \times SU(2)_m / SU(2)_k$, which include the unitary minimal models, and leave further exploration of the connection to future work.

Consider the Andrews–Baxter–Forrester model (ABF) [7]. This model is described by a square lattice on which the state variables l take the values $l =$

$1, 2, \ldots, k+1$ where k is some integer which labels the model. If l_1 and l_2 are two state variables sitting on the same bond they must obey the admissibility condition $|l_1 - l_2| = 1$. This model is in correspondence with the RCFT $SU(2)_k$ WZW model. The Boltzmann weights for a face are given in ref. [7] and depend on the temperature–like parameter p and the spectral parameter u. For $p = 0$ the model is critical. At the limit $p = 0$, $u \to i\infty$, the Boltzmann weights coincide with the braiding matrices of $SU(2)_k$ [1].

The calculation of the LSP proceeds by the corner transfer matrices method invented by Baxter [6]. We shall concentrate here on Regime III of the model where the critical point is described by the unitary minimal models [8]. The case $k = 2$ is the Ising model. The k'th model corresponds to the k'th minimal model, i.e., the coset, $SU(2)_{k-1} \times SU(2)_1 / SU(2)_k$. In regime III the following expression was found for the LSP [7]. The ground states are labeled by a pair of states, a and b, such that $|a - b| = 1$ and the pattern of the ground state is a chess board:

$$l_{i,j} = \begin{cases} a & i + j = s \bmod 2, \\ b & i + j = (1 - s) \bmod 2, \end{cases} \tag{1}$$

where $s = 0$ or 1. We can assume that $b = a + 1$. The LSP is the probability of finding the state c at the $l_{0,0}$ site, for the (a, b) ground state,

$$P(c|a, b) = \langle \delta(c, l_{0,0}) \rangle. \tag{2}$$

The corner transfer matrix calculation reduces the problem to a one dimensional configuration sum [7],

$$P(c|a, b) = \frac{\chi_c^{k-1}(q, x) q^{-\nu} \phi(c|a, b)}{\chi_a^k(q, x) \chi_1^1(q, x)}, \tag{3}$$

where ν is some power given later, eq. (44). $\chi_n^k(q, z)$ is the character of the a field

of the model $SU(2)_k$, given by,

$$\chi_a^k(q,z) = \frac{\Theta_{a+1,k+2}(q,z) - \Theta_{-a-1,k+2}(q,z)}{\Theta_{1,2}(q,z) - \Theta_{-1,2}(q,z)}, \tag{4}$$

and

$$\Theta_{j,m}(q,z) = \sum_{\gamma \in Z + \frac{j}{2m}} q^{m\gamma^2} z^{m\gamma}, \tag{5}$$

and q and x are connected to the temperature–like parameter of the lattice model, p,

$$p = e^{-\epsilon/(k+2)}, \qquad x = e^{-4\pi^2/\epsilon}, \qquad q = x^2. \tag{6}$$

The function $\phi(c|a,b)$ is the one dimensional configuration sum,

$$\phi(c|a,b) = \sum_{l_i} q^{\sum_{j=1}^n j|l_{j+2} - l_j|/4}, \tag{7}$$

where the first sum is over all admissible sequences $l_1 \sim l_2 \sim \ldots \sim l_{j+2}$, such that $l_{n+1} = a$ and $l_{n+2} = b$, $l_1 = c$, and the limit $n \to \infty$ is taken. The configuration sum was computed in ref. [7] and is expressible by characters of the minimal models,

$$\phi(c|a,b) = q^\nu \chi_{a,c}(q), \tag{8}$$

where $\chi_{a,c}(q)$ is the character of the (a,c) representation in the kth minimal model,

$$\chi_{a,c}(q) = \frac{q^{-\Delta_{a,c}}}{\prod_{j=1}^\infty (1-q^j)} \sum_{m \in Z} q^{\Delta_{a+2(k+1)m,c}} - q^{\Delta_{a+2(k+1)m,-c}}, \tag{9}$$

and

$$\Delta_{a,c} = \frac{[a(k+2) - c(k+1)]^2 - 1}{4(k+1)(k+2)}. \tag{10}$$

Our purpose is to present an alternative calculation of the configuration sum. This will provide us with the other side of the GRR identities. For this purpose

we define the following truncation of the configuration sum,

$$G_r(l_r, l_{r+1}) = \sum_{l_{r+2}, l_{r+3}, \ldots} q^{\sum_{j=l}^{\infty} j|l_{j+2}-l_j|/4}, \tag{11}$$

where the sum is over all admissible sequences $l_r \sim l_{r+1} \sim l_{r+2} \sim \ldots$, and in such a way that the (a, b) ground state is assumed, as before. Evidently,

$$G_0(c-1, c) = G_0(c+1, c) = \phi(c|a, b). \tag{12}$$

The functions $G_r(t, y)$ obey the recursion relation,

$$G_r(t, y) = \sum_{\substack{z \\ z\sim y}} q^{|z-t|/4} G_{r+1}(y, z), \tag{13}$$

which is obtained by eliminating l_r. $G_r(t, y)$ also obeys the boundary condition,

$$G_r(t, y) = \begin{cases} 1 + O(q^r) & \text{x=a and y=b} \\ O(q^r) & \text{otherwise,} \end{cases} \tag{14}$$

for large r. The recursion relation eq. (13) along with the large r limit, eq. (14), uniquely defines G_r and enables its calculation.

Next define the moments of G_r by

$$G_r(t, y) = \sum_{n=0}^{\infty} q^{nr/2} a_n(t, y). \tag{15}$$

The moments $a_n(t, y)$ obey the recursion relation,

$$a_n(t, y) = \sum_{z\sim y} q^{n/2-|z-t|/4} a_{n-\frac{1}{2}|z-t|}(y, z), \tag{16}$$

which follows from eqs. (13,15), along with the initial value,

$$a_0(t, y) = \begin{cases} 1 & \text{t=a and y=b} \\ 1 & \text{t=b and y=a} \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Again, from the recursion relation, eq. (16) one may compute, in principle at least, the moments. Our purpose is to solve these recursion relations.

We may cast the recursions relation eq. (16) in a form that will be more convenient, and also exhibits that it is indeed a recursion,

$$a_n(c-1,c) = \frac{q^{(n-1)/2}a_{n-1}(c,c+1) + q^{n-\frac{1}{2}}a_{n-1}(c-1,c-2)}{1-q^n},$$

$$a_n(c,c-1) = \frac{q^{n-\frac{1}{2}}a_{n-1}(c,c+1) + q^{(n-1)/2}a_{n-1}(c-1,c-2)}{1-q^n}, \tag{18}$$

where $c = 2, 3, \ldots, k+1$ and we define $a_n(t,y) = 0$ when t or y are out of the range $1 \leq t, y \leq k+1$.

For $k = 2$, the Ising case, the recursion relations eq. (18) are solved immediately, as they involve only one term. We easily find, for example, for the $(a,b) = (1,2)$ phase,

$$a_n(2,1) = \frac{q^{\frac{1}{2}n^2}}{(1-q)(1-q^2)\ldots(1-q^n)}, \tag{19}$$

for even n, and $a_n(2,1) = 0$ for odd n. Thus, we prove the identity,

$$\chi_{1,1}(q) = \sum_{\substack{n=0 \\ n=0 \bmod 2}}^{\infty} \frac{q^{\frac{1}{2}n^2}}{(1-q)(1-q^2)\ldots(1-q^n)}, \tag{20}$$

which appears in the list of Slater [9] of Rogers Ramanujan identities.

Let us introduce the notation

$$(q)_n = \prod_{j=1}^{n}(1-q^j), \tag{21}$$

and the q–binomial coefficients,

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{(m)_q}{(n)_q(m-n)_q}, \tag{22}$$

if $m \geq n \geq 0$, m and n integers, and $\begin{bmatrix} m \\ n \end{bmatrix} = 0$ otherwise. The q–binomial coefficients, are polynomials in q, also called Gaussian polynomials. This follows from

the fundamental recurrences they obey [10],

$$\begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n-1 \\ m \end{bmatrix} = q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}, \tag{23}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = q^{m} \begin{bmatrix} n-1 \\ m \end{bmatrix}. \tag{24}$$

We found that the solutions to the recurrences $a_n(x,y)$ are closely related to generalized Rogers–Ramanujan identities recently conjectured in ref. [11]. To describe these, following [11], introduce the sum,

$$S_p(A,Q,u) = \sum_{m \in (2Z_{\geq 0})^p + Q} q^{\frac{1}{4} m C_n m - \frac{1}{2} A m} \frac{1}{(q)_{m_1}} \prod_{s=2}^{p} \begin{bmatrix} \frac{1}{2}(m I_n + u)_s \\ m_s \end{bmatrix}, \tag{25}$$

where $A, u \in Z^p$ are vectors, $Am = \sum_{s=1}^{p} A_s m_s$, $Q \in (Z_2)^p$. C_n is the cartan matrix of A_n and $I_n = 2 - C_n$ is the incidence matrix: $(I_n)_{ab} = \delta_{a,b+1} + \delta_{a,b-1}$.

We claim that $a_n(x,y)$ is related to the above sums. Fix $p = k-1$. Denote by $M_{r,s}$ the sum

$$M_{r,s} = S_p(Q_{r,s}, e_{p+2-s}, e_r + e_{p+2-s}), \tag{26}$$

where

$$Q_{r,s} = (s-1)(e_1 + e_2 + \ldots + e_p) + (e_{r-1} + e_{r-3} + \ldots) + (e_{n+3-s} + e_{n+5-s} + \ldots), \tag{27}$$

where e_s is a unit vector, $(e_s)_x = \delta_{sx}$ and set $e_s = 0$ for $s \notin \{1, 2, \ldots, p\}$. Then for the $(a, a+1)$ phase, $a_n(x,y)$ is given by the $m_1 = n$ term in the sum for

$$a_n(c+1, c) = M_{a,c}\big|_{m_1 = n}, \tag{28}$$

$$a_n(c, c+1) = M_{p+2-a, p+2-c}\big|_{m_1 = n}, \tag{29}$$

$$a_n(p+1, p+2) = M_{a, p+2}\big|_{m_1 = n}. \tag{30}$$

The proof proceeds by inserting the expressions for a_n, eqs. (28-30), into the recursion relations, eqs. (18), and showing that they hold. The proof is incomplete

at the present as we can only show this for $k = 2, 3$, and some of the recursion relations for higher k, but not all.

The $k = 2$ case is the Ising model already mentioned. The recursion relations, eq. (18) are solved immediately, as already discussed, and we find expressions for GRR which are already known and proved. So we proceed to the first non–trivial case which is $k = 3$. The proof of the recursion relations, eq. (18) is a straight forwards application of the recurrences, eqs. (23,24). We describe as a sample identity, the cases of $c = 3$ in the second eq. (18), for the $(2, 1)$ phase,

$$a_n(3, 2)(1 - q^n) - a_{n-1}(3, 4)q^{n-\frac{1}{2}} = a_{n-1}(2, 1)q^{(n-1)/2}. \tag{31}$$

We compute the l.h.s., substituting the expressions for a_n, eqs. (28-30),

$$\begin{aligned} a_n(3,2)(1-q^n) - a_{n-1}(3,4)q^{n-\frac{1}{2}} &= \\ \sum_{\text{odd}m} \frac{q^{\frac{1}{2}(n^2+m^2-nm-m)}}{(q)_{n-1}} \left\{ \begin{bmatrix} \frac{1}{2}n+\frac{1}{2} \\ m \end{bmatrix} - q^m \begin{bmatrix} \frac{1}{2}n-\frac{1}{2} \\ m \end{bmatrix} \right\} &= \\ \frac{q^{\frac{1}{2}(n^2+m^2-nm-m)}}{(q)_{n-1}} \begin{bmatrix} \frac{1}{2}n-\frac{1}{2} \\ m-1 \end{bmatrix}&, \end{aligned} \tag{32}$$

where we used the recurrence, eq. (24). Computing the r.h.s of eqs. (31), it immediately follows that it is the same as the r.h.s of eq. (32), with the substitution of $m \to m - 1$, thus proving eq. (31). The other recurrence relations are similarly easy to prove. It is also evident that the a_n's so defined obey the initial value, eq. (17). This completes the proof that the a_n's defined by eqs. (28-30) indeed are the moments of G_r, eq. (15). From eqs. (8,15) it follows that

$$\sum_{n=0}^{\infty} a_n = \chi_{a,c}, \tag{33}$$

where $\chi_{a,c}$ is the character of the minimal model, and we have proved the GRR,

$$\chi_{a,c} = M_{a,c}, \tag{34}$$

For $k > 3$ we have only been able to prove some of the recursion relations.

Take, for example, the recurrences

$$a_n(2,1) = a_n(1,2)q^{n/2}, \tag{35}$$

$$a_n(k,k+1) = a_n(k+1,k)q^{n/2}. \tag{36}$$

These follow as a straightforward application of the definition eqs. (28-30). Similarly, the recurrences

$$a_n(1,2) = a_n(2,1)q^{n/2} + a_{n-1}(2,3)q^{(n-1)/2}, \tag{37}$$

$$a_n(k+1,k) = a_n(k,k+1)q^{n/2} + a_{n-1}(k,k-1)q^{(n-1)/2}, \tag{38}$$

follow immediately from the definition eqs. (28-30). This provides expression for some of the characters if we assume that $a_n(2,1)$ and $a_n(k,k+1)$ are indeed of the form eqs. (28-30). One can also check that the initial value eq. (17) holds. We have verified in many cases, by computer to high order, that indeed a_n is given by the expression eqs. (28-30). We believe that upon further effort the proof for $k>3$ can be completed along the lines described above.

The results described here can be generalized to other models. Consider the lattice models which correspond to IRF$(SU(2)_k,[m],[m])$, where $[m]$ stands for the representation with highest weight $(m-1)\lambda$ where λ is the fundamental weight. The states of the lattice model are, again, primary fields of SU$(2)_k$. The admissibility condition is given by the fusion rule with respect to $[m]$, $a\sim b$ if and only if

$$a = m + b \bmod 2 \text{ and } |a-b|+1 \le m \le \min(a+b-1, 2k-a-b+3). \tag{39}$$

The Boltzmann weights of the models can be obtained by the fusion procedure and are described in ref. [12].

The local state probability of this model was computed in ref. [4], and is given in terms of the one dimensional configuration sum,

$$\phi(c|a,b) = \sum_{l_i} q^{\sum_{j=1}^{n} j|l_{j+2}-l_j|/4}, \tag{40}$$

where $n \to \infty$, $l_1 = c$, $l_{n+1} = a$, $l_{n+2} = b$, and the sum is over admissible sequences, $l_1 \sim l_2 \sim l_3 \sim \ldots \sim l_{n+2}$. Notice that this configuration sum is identical to what was found before for $m = 2$, eq. (7), except for the admissibility condition which changes. The critical theory of this lattice model is the coset RCFT $O = \frac{SU(2)_{k-m+1} \times SU(2)_{m-1}}{SU(2)_k}$. It is found in ref. [4] that the configuration sum $\phi(c|a,b)$ is given by the characters of this RCFT which are the branching functions associated to this coset. Define,

$$r = \frac{1}{2}(a+b-m+1), \qquad s = \frac{1}{2}(a-b+m+1), \tag{41}$$

the configuration sum is given by,

$$\phi(c|a,b) = q^{\nu} c_{rsc}(q), \tag{42}$$

where $c_{rsc}(q)$ is the character of the RCFT O defined by,

$$\chi_r^{k-m+1}(q,z)\chi_s^{m-1}(q,z) = \sum_a c_{rsa}(q)\chi_a^k(q,z), \tag{43}$$

where $\chi_r^m(q,z)$ is the character of SU(2) at level k and isospin r, defined by eq. (4). The power ν is given by

$$\nu = \frac{1}{2}(b-c) + \gamma(r,s,c), \tag{44}$$

where

$$\gamma(r,s,c) = \frac{r^2}{4(k-m+1)} + \frac{s^2}{4(m-1)} - \frac{c^2}{4k} - \frac{1}{8}. \tag{45}$$

The local state probability in regime III is given by

$$P(a|b,c) = \frac{\chi_a^k(x^2,x)c_{rsa}(q)}{\chi_r^{k-m+1}(x^2,x)\chi_s^{m-1}(x^2,x)}, \tag{46}$$

where q and x are defined in eq. (6).

We can again try to calculate the configuration sum, $\phi(a|b,c)$ in another way by defining $G_r(l_r,l_{r+1})$ as in eq. (11),

$$G_r(l_r,l_{r+1}) = \sum_{l_{r+2},l_{r+3},\ldots} q^{\sum_{j=l}^{\infty} j|l_{j+2}-l_j|/4}, \tag{47}$$

where the sum is over all admissible sequences $l_r \sim l_{r+1} \sim l_{r+2} \sim \ldots$, and is taken in the (a,b) phase. As before,

$$G_0(d,c) = \phi(c|a,b), \tag{48}$$

for all d such that $d \sim c$. We have the same recursion relation for $G_r(t,y)$ as before, eq. (13). We define the moments of $G_r(t,y)$, $a_n(t,y)$, as before, eq. (15), and they obey a similar recursion relation,

$$a_n(t,y) = \sum_{z\sim y} q^{n/2-|z-t|/4} a_{n-\frac{1}{2}|z-t|}(y,z), \tag{49}$$

along with the initial value, eq. (17).

The complete solution for the moments $a_n(t,y)$ is not known. However, we can obtain parts of the solution, and through that new GRR identities. Our starting point is the conjecture proposed in ref. [11] for the character of the identity in the RCFT O. This is given by the expression,

$$M = \sum_{l_1,l_2,\ldots,l_n\in(2Z_{\geq 0})^n} \frac{q^{\frac{1}{4}lC_nl}}{(q)_{l_{m-1}}} \prod_{\substack{a=1\\ a\neq m-1}}^{n} \begin{bmatrix} \frac{1}{2}(l_{a-1}+l_{a+1}) \\ l_a \end{bmatrix}, \tag{50}$$

where C_n is the Cartan matrix of A_n. We find that $a_n(m,1)$ in the $(1,m)$ phase is

given by the $l_{m-1} = n$'th term of eq. (50),

$$a_n(m,1) = M\Big|_{l_{m-1}=n}, \tag{51}$$

for even n, and is zero for odd n. This is consistent with the fact that the identity character is given by

$$c_{1,1,1}(q) = \sum_{n=0}^{\infty} a_n = M. \tag{52}$$

We checked this in various examples by computer to high order. Accepting this conjecture, we can immediately find new GRR by utilizing the recursion relations eq. (49). From the recursion relation,

$$a_n(m,1) = a_n(1,m)q^{n/2}, \tag{53}$$

we get an expression for $a_n(1,m)$,

$$a_n(1,m) = q^{-n/2}M\big|_{l_{m-1}=n}, \tag{54}$$

which leads to the character identity,

$$c_{1,1,m} = \sum_{l_1,l_2,\ldots,l_n\in(2Z_{\geq 0})^n} \frac{q^{\frac{1}{4}lC_nl-\frac{1}{2}l_{m-1}}}{(q)_{l_{m-1}}} \prod_{\substack{a=1\\a\neq m-1}}^{n} \begin{bmatrix} \frac{1}{2}(l_{a-1}+l_{a+1}) \\ l_a \end{bmatrix}. \tag{55}$$

As an example, we have computed characters for $m = 3$ and $k = 4$. This is the second minimal model for the super-Virasoro algebra. The states are labeled by $a \in \{1, 2, \ldots, 5\}$. We find for the moments $a_n(x,y)$ in the $(3,1)+(1,3)$ phase,

$$a_n(3,1) = \sum_{m_3\ \text{even},m_1} \frac{q^{\frac{1}{2}(m_1^2+n^2+m_3^2-nm_1-nm_3)}}{(q)_n} \begin{bmatrix} \frac{1}{2}n \\ m_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}n \\ m_3 \end{bmatrix}, \tag{56}$$

for even n; $a_n(3,1)=0$ for odd n.

$$a_n(1,3)=\sum_{m_3\ \text{even},m_1}\frac{q^{\frac{1}{2}(m_1^2+n^2+m_3^2-nm_1-nm_3-n)}}{(q)_n}\begin{bmatrix}\frac{1}{2}n\\ m_1\end{bmatrix}\begin{bmatrix}\frac{1}{2}n\\ m_3\end{bmatrix}, \tag{57}$$

for even n; $a_n(1,3)=0$ for odd n.

$$a_n(3,5)=\sum_{m_3\ \text{odd},m_1}\frac{q^{\frac{1}{2}(m_1^2+n^2+m_3^2-nm_1-nm_3)}}{(q)_n}\begin{bmatrix}\frac{1}{2}n\\ m_1\end{bmatrix}\begin{bmatrix}\frac{1}{2}n\\ m_3\end{bmatrix}, \tag{58}$$

for even n. $a_n(3,5)=0$ for odd n.

$$a_n(5,3)=\sum_{m_3\ \text{odd},m_1}\frac{q^{\frac{1}{2}(m_1^2+n^2+m_3^2-nm_1-nm_2-n)}}{(q)_n}\begin{bmatrix}\frac{1}{2}n\\ m_1\end{bmatrix}\begin{bmatrix}\frac{1}{2}n\\ m_3\end{bmatrix}, \tag{59}$$

for even n.

$$a_n(3,3)=\sum_{m_1,m_3}\frac{q^{\frac{1}{2}(m_1^2+n^2+m_3^2-nm_1-nm_3-n+m_1+m_3)}}{(1-q^{n/2})(q)_{n-1}}\begin{bmatrix}\frac{1}{2}(n-1)\\ m_1\end{bmatrix}\begin{bmatrix}\frac{1}{2}(n-1)\\ m_3\end{bmatrix}, \tag{60}$$

for odd n. In the $(3,3)$ phase we find,

$$a_n(3,1)=\sum_{m_1,m_3\ \text{odd}}\frac{q^{\frac{1}{2}(m_1^2+n^2+m_3^2-nm_1-nm_3)}}{(q)_n}\begin{bmatrix}\frac{1}{2}(n+1)\\ m_1\end{bmatrix}\begin{bmatrix}\frac{1}{2}(n+1)\\ m_3\end{bmatrix}, \tag{61}$$

for odd n; $a_n(3,1)=0$ for even n.

$$a_n(1,3)=\sum_{m_1,m_3\ \text{odd}}\frac{q^{\frac{1}{2}(m_1^2+n^2+m_3^2-nm_1-nm_3-n)}}{(q)_n}\begin{bmatrix}\frac{1}{2}(n+1)\\ m_1\end{bmatrix}\begin{bmatrix}\frac{1}{2}(n+1)\\ m_3\end{bmatrix}, \tag{62}$$

for odd n; $a_n(1,3)=0$ for even n.

$$a_n(3,3)=\sum_{m_1,m_3\ \text{odd}}\frac{2q^{\frac{1}{2}(m_1^2+n^2+m_3^2-nm_1-nm_3-n+m_1+m_3)}}{(1-q^{n/2})(q)_{n-1}}\begin{bmatrix}\frac{1}{2}n\\ m_1\end{bmatrix}\begin{bmatrix}\frac{1}{2}n\\ m_3\end{bmatrix}, \tag{63}$$

for even n; $a_0(3,3)=1$; $a_n(3,3)=0$ for odd n. Using the symmetry property $a_n(x,y)$ in the (a,b) phase is the same as $a_n(k+2-x,k+2-y)$ in the $(k+2-a,k+2-b)$ phase, we get all the moments of the configuration sum for x,y,a,b even (the even sector).

We find nice expressions for most of the sums in the odd sector, as well. In the $(2,2)$ ground state we have,

$$a_n(2,4) = \sum_{m_1 \text{ even}, m_3 \text{ odd}} \frac{q^{\frac{1}{2}(m_1^2+n^2+m_3^2-nm_1-nm_3-m_1)}}{(q)_n} \begin{bmatrix} \frac{1}{2}(n+1) \\ m_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(n+1) \\ m_3 \end{bmatrix}, \quad (64)$$

for even n.

$$a_n(2,2) = \sum_{m_3 \text{ odd}, m_1 \text{ even}} \frac{q^{\frac{1}{2}(m_1^2+n^2+m_3^2-nm_1-nm_3+m_3-n)}}{(1-q^{n/2})(q)_{n-1}} \begin{bmatrix} \frac{1}{2}n \\ m_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}n \\ m_3 \end{bmatrix}, \quad (65)$$

for even n. In the $(2,4)+(4,2)$ ground state we find,

$$a_n(2,2) = \sum_{m_2 \text{ odd}, m_1} \frac{q^{\frac{1}{2}(m_1^2+n^2+m_3^2-nm_1-nm_3-m_1)}}{(q)_n} \begin{bmatrix} \frac{1}{2}(n+1) \\ m_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(n+1) \\ m_3 \end{bmatrix}, \quad (66)$$

for odd n.

$$a_n(2,4) = \sum_{m_3 \text{ odd}, m_1} \frac{q^{\frac{1}{2}(m_1^2+n^2+m_3^2-nm_1-nm_3+n-m_1-m_3)}}{(1+q^{(n+1)/2})(q)_n} \begin{bmatrix} \frac{1}{2}n+1 \\ m_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}n+1 \\ m_3 \end{bmatrix}, \quad (67)$$

for even n.

By summing over $a_n(t,y)$ we find new GRR character identities. Since $G_0(t,y) = \sum_{n=0}^{\infty} a_n(t,y) = c_{rsa}(q)$, we get the identities,

$$c_{111} + c_{131} = G_0(3,1), \quad (68)$$

$$c_{113} + c_{133} = G_0(1,3) = G_0(3,3) = G_0(5,3), \quad (69)$$

$$c_{115} + c_{135} = G_0(3,5), \quad (70)$$

in the $(1,3)$ phase. In the $(3,3)$ phase we find,

$$c_{221} = G_0(3,1), \quad (71)$$

$$c_{223} = G_0(1,3) = G_0(3,3). \tag{72}$$

In the $(2,2)$ phase we find,

$$c_{124} = G_0(2,4), \tag{73}$$

$$c_{122} = G_0(2,2). \tag{74}$$

In the $(2,4)$ phase we find

$$c_{212} + c_{232} = G_0(2,2) = G_0(2,4). \tag{75}$$

This completes the description of new GRR associated with this theory. We expect that this can be generalized to all m and all k.

We described here the appearance of generalized Rogers–Ramanujan identities in the expressions for the local state probability in solvable lattice models. We believe that this correspondence is quite general and could be extended to many other such models. It is hoped that this work will be of help in understanding solvable lattice models, their relationship to RCFT, and generalized Rogers–Ramanujan identities.

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REFERENCES

1. D. Gepner, Caltech preprint, CALT-68-1825, November 1992
2. D. Gepner, in "Non perturbative methods in field theory", Irvine 1987, Conference proceedings, Nucl. Phys. A1 [PS] (1987) 105
3. E. Date, M. Jimbo, T. Miwa and M. Okado, Phys. Rev. B35 (1987) 2105
4. E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Nucl. Phys. B290 [FS20] (1987) 231
5. R. Kedem, T.R. Klassen, B.M. McCoy and E. Melzer, Phys. Lett. B304 (1993) 263; B.L. Feigin, T. Nakanishi and H. Ooguri, Int. J. Mod. Phys. A7, Suppl. 1A (1992) 217; W. Nahm, A. Recknagel and M. Terhoeven, Mod. Phys. Lett. A8 (1993) 1835; M. Tehoeven, Bonn preprint BONN-HE-92-36, hepth/9111120; A. Kuniba, T. Nakanishi and J. Suzuki, Mod. Phys. Lett. A8 (1993) 1649
6. R.J. Baxter, Exactly solved models in statistical mechanics, Academic Press, London, 1982
7. G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193
8. D.A. Huse, Phys. Rev. D30 (1984) 3908
9. L.J. Slater, Proc. London Math. Soc. 54 (1953) 147
10. G.E. Andrews, The theory of partitions, (Addison–Wesley, London, 1976
11. R. Kedem, T.R. Klassen, B.M. McCoy and E. Melzer, Phys. Lett. B307 (1993) 68
12. E. Date, M. Jimbo, T. Miwa and M. Okado, Lett. Math. Phys. 12 (1986) 209